Dirac spinors in solenoidal field and self adjoint extensions of its Hamiltonian

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We discuss Dirac equation (DE) and its solution in presence of solenoid (infinitely long) field in (3+1) dimensions. Starting with a very restricted domain for the Hamiltonian, we show that a 1-parameter family of self adjoint extensions (SAE) are necessary to make sure the correct evolution of the Dirac spinors. Within the extended domain bound state (BS) and scattering state (SS) solutions are obtained. We argue that the existence of bound state in such system is basically due the breaking of classical scaling symmetry by the quantization procedure. A remarkable effect of the scaling anomaly is that it puts an open bound on both sides of the Dirac sea, i.e., $E \in (-M, M)$ for $\nu^2[0,1)$! We also study the issue of relationship between scattering state and bound state in the region $\nu^2 \in [0,1)$ and recovered the bound state solution and eigenvalue from the scattering state solution.

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I. INTRODUCTION

The problem of a Dirac particle in magnetic fields background has been studied extensively in literature [1, 2, 3, 4]. The solution in a uniform field background [1] is known for a long time. For a solenoidal background, the Dirac equation has been solved in (2+1) dimensions [2, 3, 4]. It shows that a self-adjoint extension of the Dirac Hamiltonian is necessary to obtain the solutions. In (3+1) dimensions, self-adjoint extension of the Dirac Hamiltonian has also been studied [5], but it involves a δ-sphere potential rather than solenoid interaction. In Ref. [6], self-adjoint extension of Dirac Hamiltonian in solenoid field and an uniform magnetic field B has be studied in both (2+1) and (3+1) dimensions elegantly.

The consequences of considering self-adjoint extensions for the problem of Dirac particle in solenoidal field is recognized in Ref. [7]. Because SAE [8, 9, 10] is a method which allows to consistently build the all possible boundary conditions under which the Hamiltonian is self-adjoint. It is therefore expected that [2] the results would be function of the boundary conditions, characterized by parameters.

In this paper, we study the solution of the Dirac equation in only infinitely long solenoid field in $(3+1)$ dimensions. Because, the magnetic field of an infinitely long solenoid is immensely important for various reasons. For example, it is used to study the Aharanov-Bohm [11] effect. Similarity with this field also helps one to understand the physics of cosmic strings [12]. Although, in principle problem in Ref. [6] should reduce to the problem involving only solenoid field by making the uniform magnetic field $B = 0$. But it is a nontrivial task. So it is important to study Dirac free Hamiltonian in only solenoid field background in (3+1) dimensions and solve it with the most generalized boundary conditions, such that the Hamiltonian is self-adjoint. Some comments about the difference between method of [6] and ours will be made in Sec. VII.

In order to get the most generalized boundary condition for our problem, we start with a very restricted domain. Obviously the Hamiltonian is not self-adjoint in that domain. We then go for a self-adjoint extensions of the Hamiltonian by using the von Neumann's method [15]. We map the problem from the Hilbert space $L_2(drd\phi dz; \mathcal{C}^4)$ to $L_2(drd\phi dz; \mathcal{C})$ keeping in mind that we seek self-adjointness of the whole (4×4) Dirac Hamiltonian in solenoid field background. In the Hilbert space $L_2(drd\phi dz; \mathcal{C})$ we get a well known one dimensional Schrödinger eigenvalue problem of inverse square interaction. Inverse square interaction is known to be classically scale invariant. But, quantization of this system shows that the classical scale symmetry is broken for some values of the self-adjoint extension parameter and for a very limited range $-1/4 \le g < 3/4$ of the coupling constant g of the inverse square interaction. Breaking of this scaling symmetry by quantization is known as scaling anomaly. The indication of scaling anomaly in the system is manifested through the formation of bound state, which is supposed to be absent if scaling symmetry is present even after quantization. In case of Dirac particle in solenoid field, this scaling anomaly is also responsible for the formation of bound state. It is also possible to comment on the bounds (upper and lower bounds) of the bound state energy of the Dirac particle on solenoid filed background. By taking appropriate limit, it is even possible to find the upper and lower bounds of the energy of the free Dirac particle.

The paper is organized as follows: In Sec. II, we state the problem of Dirac equation (DE) in solenoid field background. In Sec. III, we state the usual symmetric boundary condition using regularity and square integrability argument and we perform the required self adjoint extension. In Sec. IV, we discuss the bound states of the radial Hamiltonian and we get the bound state condition. In Sec. V, we discuss the scattering states of the radial

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Hamiltonian and get the corresponding condition for it. In Sec. VI, we discuss the scaling anomaly present in the system. We discuss in Sec. VII.

II. SOLUTIONS OF THE DE IN SOLENOID FIELD

We consider a Dirac particle of mass M and charge eQ in solenoid field background. Due to the cylindrical symmetry of the problem it is better to use the cylindrical polar coordinate system, where the spatial coordinates are denoted by r, ϕ, z. In any orthogonal coordinate system, the Dirac equation for a particle with charge eQ and mass M can be written as Ref. [13]

$$\Big[\gamma^\mu \left(i\hat{D}_\mu - eQA_\mu\right) - M \\ +i\sum_i \gamma_i \left[\frac{1}{2}\hat{D}_i \log\left(h_1 h_2 h_3/h_i\right)\right]\Big]\Psi_r \ = 0, \tag{1}$$

where $h_i's$ are scale factors of the corresponding coordinate satem. In our case since we are using cylindrical coordinate system, $h_1 = 1, h_2 = r, h_3 = 1$. the derivative $\hat{D}_\mu = (h_\mu)^{-1}\partial_\mu$, where no summation over μ is implied here and A_μ is the infinitely long solenoid vector potential, $\vec{A} = \frac{\alpha}{r}\hat{\phi}$. One can make a conformal transformation [13], which reduces (1) to simpler form with zero spin connection as [13],

$$\left[\gamma^\mu\left(i\hat{D}_\mu - eQA_\mu\right) - M\right]\Psi(t,r,\phi,z) = 0, \tag{2}$$

where the relation between Ψ_r and Ψ is $\Psi_r = (1/\sqrt{r})\exp(-\frac{1}{2}i\phi\Sigma_3)\Psi$. Σ_3 is the generator for rotation along z-axis [13]. The normalization condition of the new wave-function Ψ is

$$\int drd\phi dz\ \Psi^\dagger(t,r,\phi,z)\Psi(t,r,\phi,z) = 1. \tag{3}$$

To solve the Dirac equation (2) we take the trial solution of the Dirac equation of the form

$$\Psi(t,r,\phi,z) = e^{-iEt}e^{-im\phi}e^{-ip_z z}\begin{pmatrix}\phi(r)\\ \chi(r)\end{pmatrix}, \tag{4}$$

where ϕ and χ are 2-component objects. We use the Pauli-Dirac representation of the Dirac matrices

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{5}$$

where each block represents a 2×2 matrix, and σ_is are the Pauli matrices. Note that the same γ^μ matrices are used which we use in Cartesian co-ordinate system. For detail discussion about the Dirac equation in cylindrical co-ordinates and the form of the γ^μ matrices for cylindrical co-ordinate system see Ref. [13]. Multiplying Eq. (2) by γ^0 from left, and using Eqs. (4) and (5) we get

$$\begin{pmatrix} E-M & \mathcal{A} \\ \mathcal{A} & E+M\end{pmatrix}\begin{pmatrix}\phi(r)\\ \chi(r)\end{pmatrix} = 0, \tag{6}$$

where $\mathcal{A} = \boldsymbol{\sigma}.\left(i\hat{D}_\mu - eQA_\mu\right) = i\sigma^1\partial_r + \frac{m-eQ\alpha}{r}\sigma^2 + \sigma^3 p_z$. Eq. (6) is divided into two coupled equations

$$\phi(r) \ = \ \frac{\left[i\sigma^1\partial_r + \frac{m-eQ\alpha}{r}\sigma^2 + \sigma^3 p_z\right]}{E-M}\chi(r), \tag{7}$$

$$\chi(r) \ = \ \frac{\left[i\sigma^1\partial_r + \frac{m-eQ\alpha}{r}\sigma^2 + \sigma^3 p_z\right]}{E+M}\phi(r). \tag{8}$$

Eliminating $\chi(r)$, we obtain

$$\phi(r) \ = \ \frac{\left[i\sigma^1\partial_r + \frac{m-eQ\alpha}{r}\sigma^2 + \sigma^3 p_z\right]^2}{E^2-M^2}\phi(r). \tag{9}$$

There will be two independent solutions for $\phi(r)$, which can be taken, without any loss of generality, to be the eigen-states of σ_z with eigenvalues $s = \pm 1$. This means that we can choose two independent solutions of the form

$$\phi(r) = \begin{pmatrix}F_+(r)\\ 0\end{pmatrix}, \qquad \phi(r) = \begin{pmatrix}0\\ F_-(r)\end{pmatrix}. \tag{10}$$

Since $\sigma^3\phi(r) = s\phi(r)$, Eq. (9) becomes

$$\phi(r)\mid_{s=\pm1} = \frac{-\frac{d^2}{dr^2} + \frac{(m-eQ\alpha\mp1/2)^2-1/4}{r^2} + p_z^2}{E^2-M^2}\phi(r) \tag{11}$$

For $s = +1$, using (10) in (11), the differential equation satisfied by F_+ is

$$\frac{d^2F_+}{dr^2} + \left(\lambda^2 - \frac{\nu^2-1/4}{r^2}\right)F_+ = 0, \tag{12}$$

where

$$\lambda = (E^2-M^2-p_z^2)^{1/2}, \nu = m - eQ\alpha - 1/2. \tag{13}$$

It should be noted that, (12) can be considered as a well known one dimensional Schrödinger eigen-value equation with inverse square interaction [8, 9, 10]. It shows classical scale symmetry and scaling anomaly, which will be discussed separately in Sec. VI. Now we seek the solution of this equation, which is of the form $F_+ = r^{\frac{1}{2}}\mathscr{C}_\nu(\lambda r)$. Where $\mathscr{C}$ denotes J, Y, H^1, H^2 or any linear combination of these functions with constant coefficients. From (8), we find the two lower components and write the spinor as

$$U_+(r,p_z) \equiv \begin{pmatrix} r^{\frac{1}{2}}\mathscr{C}_\nu(\lambda r) \\ 0 \\ \dfrac{p_z}{E+M}r^{\frac{1}{2}}\mathscr{C}_\nu(\lambda r) \\ \dfrac{i\lambda}{E+M}r^{\frac{1}{2}}\mathscr{C}_{\nu+1}(\lambda r)\end{pmatrix}, \tag{14}$$

where the normalization has not been specified. For $s = -1$, we get the same Bessel differential equation (12), with ν replaced by $\nu + 1$. The solution, which can be obtained similarly for this case also, is of the form

$$U_-(r, p_z) \equiv \begin{pmatrix} 0 \\ r^{\frac{1}{2}} \mathscr{C}_{\nu+1}(\lambda r) \\ \dfrac{-i\lambda}{E+M} r^{\frac{1}{2}} \mathscr{C}_\nu(\lambda r) \\ \dfrac{-p_z}{E+M} r^{\frac{1}{2}} \mathscr{C}_{\nu+1}(\lambda r) \end{pmatrix}, \tag{15}$$

A similar procedure can be adopted for negative frequency spinors. In this case, it is easier to start with the two lower components first and then find the upper components. The negative energy spinors are found to be

$$V_+(r, p_z) \equiv \begin{pmatrix} \dfrac{p_z}{E+M} r^{\frac{1}{2}} \mathscr{C}_\nu(\lambda r) \\ \dfrac{i\lambda}{E+M} r^{\frac{1}{2}} \mathscr{C}_{\nu+1}(\lambda r) \\ r^{\frac{1}{2}} \mathscr{C}_\nu(\lambda r) \\ 0 \end{pmatrix}, \tag{16}$$

$$V_-(r, p_z) \equiv \begin{pmatrix} \dfrac{-i\lambda}{E+M} r^{\frac{1}{2}} \mathscr{C}_\nu(\lambda r) \\ \dfrac{-p_z}{E+M} r^{\frac{1}{2}} \mathscr{C}_{\nu+1}(\lambda r) \\ 0 \\ r^{\frac{1}{2}} \mathscr{C}_{\nu+1}(\lambda r) \end{pmatrix}. \tag{17}$$

III. SAE OF RADIAL HAMILTONIAN

We now discuss the self-adjointness problem of our system. Therefore we need to know the radial eigenvalue problem, which can be obtained from (2). It can be shown that the radial eigenvalue equation is of the following form

$$H(r)S(r) = ES(r), \tag{18}$$

where the radial Hamiltonian and the eigenfunction are given by

$$H(r) = \begin{pmatrix} M & -\mathcal{A} \\ -\mathcal{A} & -M \end{pmatrix}, S(r) = \begin{pmatrix} \phi(r) \\ \chi(r) \end{pmatrix}, \tag{19}$$

where $\mathcal{A} = i\sigma^1 \partial_r + \frac{m - eQ\alpha}{r}\sigma^2 + \sigma^3 p_z$. The differential operator $H(r)$ is symmetric over the domain $D(H) = \psi(r)$, where $\psi(r) \in L_2(dr; C^4)$ and $\psi(0) = 0$. This means that for $\psi_1, \psi_2 \in D(H)$, the radial Hamiltonian $H(r)$ satisfies the condition

$$\int_0^\infty dr \psi_1^\dagger(r) H(r) \psi_2(r) = \int_0^\infty dr \left[H(r)\psi_1(r)\right]^\dagger \psi_2(r) \tag{20}$$

However, in domain $D(H)$ the radial Hamiltonian $H(r)$ is not self-adjoint. A symmetric Hamiltonian is self-adjoint if its domain coincides with that of the domain of its adjoint, i.e, $D(H) = D(H^\dagger)$. The condition at the origin makes the Hamiltonian non self-adjoint. The domain of the adjoint Hamiltonian $H^\dagger(r)$ is $D(H^\dagger) = \psi(r)$, where $\psi(r) \in L_2(dr; C^4)$. We see that $D(H) \neq D(H^\dagger)$, indicating that $H(r)$ is not self-adjoint. To make the Hamiltonian self adjoint [15] we use von Neumann's method of deficiency indices. It requires the construction of eigen-space $D^\pm$ of $H^\dagger$ with eigenvalue $\pm iM$ ($M \neq 0$ is inserted for dimensional reason). The up spinors (particle state) for the eigen-space $D^\pm$ are

$$\phi^\pm = N \begin{pmatrix} r^{\frac{1}{2}} H_\nu^{1,2}(e^{\pm i\frac{\pi}{2}} \lambda_1 r) \\ 0 \\ \dfrac{p_z}{M(1 \pm i)} r^{\frac{1}{2}} H_\nu^{1,2}(e^{\pm i\frac{\pi}{2}} \lambda_1 r) \\ \dfrac{ie^{\pm i\frac{\pi}{2}}\lambda_1}{M(1 \pm i)} r^{\frac{1}{2}} H_{\nu+1}^{1,2}(e^{\pm i\frac{\pi}{2}} \lambda_1 r) \end{pmatrix}, \tag{21}$$

where $\lambda_1 = \left(2M^2 + p_z^2\right)^{\frac{1}{2}}$ and N is the normalization constant. Here we have written $\phi^\pm$ for up spinor (particle state) only. Similarly we will get $\phi^\pm$ for all other spinors. Since we will do our calculations for spin up particle state only, (21) is sufficient for us. Looking at the asymptotic form of the Hankel functions

$$H_\nu^1(z) \to \left[\frac{2}{\pi z}\right]^{1/2} e^{i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}, \quad \text{for } (-\pi < \arg z < 2\pi) \tag{22}$$

$$H_\nu^2(z) \to \left[\frac{2}{\pi z}\right]^{1/2} e^{-i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}, \quad \text{for } (-2\pi < \arg z < \pi) \tag{23}$$

we find that the upper end of the integrals for evaluating the norms of $\phi^\pm$ are finite for any ν. However, near $r = 0$, the Hankel function behavior can be found from the short distance behavior of the Bessel function

$$J_\nu(z) \to \frac{z^\nu}{2^\nu \Gamma(1+\nu)}, \quad (\nu \neq -1, -2, -3, ...) \tag{24}$$

Considering all components of the spinor of (21), we find that $\phi^\pm$ are square integrable only in the interval

$$0 \leq \nu^2 < 1. \tag{25}$$

Since beyond this range there is no square integrable solutions $\phi^\pm$, the deficiency indices, which are dimensions of the eigen-space $D^\pm$,

$$n_\pm = dim(D^\pm), \tag{26}$$

are zero , i.e,

$$n_+ = n_- = 0. \tag{27}$$

The closure of the operator $H(r)$ is the self-adjoint extension for the case (27). We therefore concentrate for the interval (25) to carry out self-adjoint extensions. The existence of complex eigenvalues for $H^\dagger(r)$ emphasizes the lack of self-adjointness. The self-adjoint extensions of $H(r)$ are labeled by the isometries $D^+ \to D^-$, which can be parameterized by

$$\phi^+(r) \to e^{i\omega}\phi^-(r) \tag{28}$$

The correct domain (It is better to call projection of the domain on spin up particle state direction rather than only domain because we are concentrating on spin up particle state only here) for the self-adjoint extension $H^\omega(r)$ of $H(r)$ is then given by

$$D^\omega(H^\omega) \equiv D(H) + \phi^+(r) + e^{i\omega}\phi^-(r)\,, \tag{29}$$

where $\omega \in \mathbb{R}(\mathrm{mod} 2\pi)$. In the next section we find out the bound state solutions using the domain $D^\omega(H^\omega)$.

IV. SOLUTIONS OF RADIAL HAMILTONIAN

From the trial solution (4) it is clear that the spinors along the z-direction is free, as it should be, because there is no constraint in the z-direction. We therefore try to investigate whether the spinors are bound in the radial direction. From now on bound state means bound in the radial direction. Throughout our calculation, we will use the spin up state. Calculation for all other spinor states are similar. From the general spin up state (14) it is easy to see that it serves as a square integrable spin up state if we use $\mathscr{C}_\nu(\lambda r) = H^1_\nu(\lambda r)$ and if $\lambda = (E^2 - M^2 - p_z^2)^{\frac{1}{2}} = iq$, where q is real positive. Similarly we could have used $\mathscr{C}_\nu(\lambda r) = H^2_\nu(\lambda r)$ as square integrable function if we took $\lambda = (E^2 - M^2 - p_z^2)^{\frac{1}{2}} = -iq$, where q is real positive. We will use Hankel function of the first kind H^1_ν to express bound state solution. The bound state spinor of spin up particle is then found to be

$$U_+(r,p_z) = B\sqrt{E+M}\begin{pmatrix} r^{\frac{1}{2}}H^1_\nu(\lambda r) \\ 0 \\ \dfrac{p_z}{E+M}r^{\frac{1}{2}}H^1_\nu(\lambda r) \\ \dfrac{i\lambda}{E+M}r^{\frac{1}{2}}H^1_{\nu+1}(\lambda r)\end{pmatrix}, \tag{30}$$

where B is the normalization constant. To find out bound state eigenvalue we have to match the limiting value $r \to 0$ of spinor (30) with the limiting value $r \to 0$ of the domain (29). For the sake of simplicity, we set $p_z = 0$ before matching at the origin. The relevant range of ν is given in (25). In this range, the leading r-behavior of the domain (29) for small r is given by

$$\psi(r) + \phi^+(r) + e^{i\omega}\phi^-(r) \to \begin{pmatrix} A(\lambda_1)r^{\nu+1/2} \\ 0 \\ 0 \\ D(\lambda_1)r^{-\nu-1/2}\end{pmatrix}, \tag{31}$$

where $A(\lambda_1) = N\frac{i}{\sin\nu\pi}\frac{\lambda_1^\nu}{2^\nu\Gamma(1+\nu)}\left[e^{-\frac{\pi\nu i}{2}} - e^{i\omega}e^{\frac{\pi\nu i}{2}}\right]$, $D(\lambda_1) = -Ni\frac{i}{\sin(\nu+1)\pi}\frac{\lambda_1^{-\nu-1}}{2^{-\nu-1}\Gamma(-\nu)}\left[e^{-\frac{\pi\nu i}{2}-\frac{\pi i}{4}} - e^{i\omega}e^{\frac{\pi\nu i}{2}+\frac{\pi i}{4}}\right]$, and the leading term of the spinor (30) is

$$U_+(r,p_z) \to \begin{pmatrix} \tilde{A}(\lambda)r^{\nu+1/2} \\ 0 \\ 0 \\ \tilde{D}(\lambda)r^{-\nu-1/2}\end{pmatrix}, \tag{32}$$

where $\tilde{A}(\lambda) = B\sqrt{E+M}\frac{i}{\sin\nu\pi}\frac{\lambda^\nu}{2^\nu\Gamma(1+\nu)}e^{-\pi\nu i}$, $\tilde{D}(\lambda) = -Bi\sqrt{E-M}\frac{i}{\sin(\nu+1)\pi}\frac{\lambda^{-\nu-1}}{2^{-\nu-1}\Gamma(-\nu)}$. Since $U(r,p_z) \in D^\omega(H^\omega)$, the coefficients of leading powers of r in (31) and (32) must match and this gives the eigenvalue equation

$$\frac{\left(1+\frac{E}{M}\right)^{\nu+1}}{\left(1-\frac{E}{M}\right)^{-\nu}} = -2^{\nu+1/2}\frac{\sin\left(\omega/2+\pi\nu/2\right)}{\sin\left(\omega/2+\pi\nu/2+\pi/4\right)} \tag{33}$$

The left hand side of (33) is positive. So right hand side should be positive and to ensure that we impose the condition $\cot(\frac{\omega}{2}+\frac{\pi\nu}{2}) < -1$, which is the bound state condition. Similarly we may get all other spinors for the bound states. We move to the next section for the discussion of scattering state (SS) solutions.

V. SS SOLUTIONS OF RADIAL HAMILTONIAN

The scattering state spinor of spin up particle is

$$U_+(r,p_z) = B\sqrt{E+M}\begin{pmatrix} r^{\frac{1}{2}}A_\nu \\ 0 \\ \dfrac{p_z}{E+M}r^{\frac{1}{2}}A_\nu \\ \dfrac{i\lambda}{E+M}r^{\frac{1}{2}}A_{\nu+1}\end{pmatrix}, \tag{34}$$

where B is the normalization constant, $A_\nu = [a(\lambda)J_\nu(\lambda r) + b(\lambda)J_{-\nu}(\lambda r)]$ and $A_{\nu+1} = \left[\tilde{a}(\lambda)J_{\nu+1}(\lambda r) - \tilde{b}(\lambda)J_{-\nu-1}(\lambda r)\right]$, $a(\lambda)$, $b(\lambda)$, $\tilde{a}(\lambda)$ and $\tilde{b}(\lambda)$ are constant coefficients. To find out eigenvalue for the scattering state we have to match the

limiting value $r \to 0$ of the spinor (34) with (29). For simplicity of calculation we set $p_z = 0$ in (34) and (29) before matching. In the limit $r \to 0$, the spinor (34) looks like (32) but now the coefficients of different powers of r are, $\tilde{A}(\lambda) = B\sqrt{E+M}a(\lambda)\frac{\lambda^{\nu}}{2^{\nu}\Gamma(1+\nu)}$, $\tilde{D}(\lambda) = -Bi\sqrt{E-M}\tilde{b}(\lambda)\frac{\lambda^{-\nu-1}}{2^{-\nu-1}\Gamma(-\nu)}$ and the limit $r \to 0$ of (29) is given in (31). Again equating the respective coefficients and comparing between them we get the eigenvalue equation

$$\frac{(E+M)^{1/2}}{(E-M)^{1/2}}\frac{a(\lambda)}{\tilde{b}(\lambda)}\left(\frac{\lambda}{\lambda_1}\right)^{2\nu+1} = -\frac{\sin(\omega/2+\pi\nu/2)}{\sin(\omega/2+\pi\nu/2+\pi/4)} \tag{35}$$

The right hand side of (35) has to be positive in order to get scattering state solutions. Similarly we may get all other spinors for scattering states. We now want to show the relation between scattering state and bound state [16, 17] for completeness of our calculation. To calculate that we expand (34) in the limit $r \to \infty$. The leading term in the asymptotic expansion of $U_+(r, p_z)$, without normalization is given by

$$U_+(r, p_z) \to \begin{pmatrix} A(\lambda)e^{i\lambda r} + B(\lambda)e^{-i\lambda r} \\ 0 \\ \frac{p_z}{E+M}A(\lambda)e^{i\lambda r} + \frac{p_z}{E+M}B(\lambda)e^{-i\lambda r} \\ C(\lambda)e^{i\lambda r} + D(\lambda)e^{-i\lambda r} \end{pmatrix} \tag{36}$$

where different coefficients are found to be $A(\lambda) = \sqrt{\frac{E+M}{2\pi\lambda}}\left[a(\lambda)e^{-i\frac{\pi}{2}(\nu+1/2)} + b(\lambda)e^{i\frac{\pi}{2}(\nu-1/2)}\right]$, $B(\lambda) = \sqrt{\frac{E+M}{2\pi\lambda}}\left[a(\lambda)e^{i\frac{\pi}{2}(\nu+1/2)} + b(\lambda)e^{-i\frac{\pi}{2}(\nu-1/2)}\right]$, $C(\lambda) = i\sqrt{\frac{E-M}{2\pi\lambda}}\left[\tilde{a}(\lambda)e^{-i\frac{\pi}{2}(\nu+3/2)} - \tilde{b}(\lambda)e^{i\frac{\pi}{2}(\nu+1/2)}\right]$, $D(\lambda) = i\sqrt{\frac{E-M}{2\pi\lambda}}\left[\tilde{a}(\lambda)e^{i\frac{\pi}{2}(\nu+3/2)} - \tilde{b}(\lambda)e^{-i\frac{\pi}{2}(\nu+1/2)}\right]$. Now it is easy to see from the asymptotic expansion (36) that $e^{-i\lambda r}$ blows up on the positive imaginary λ-axis but the other part $e^{i\lambda r}$ decays exponentially there. So it reasonable to set the coefficients $B(\lambda)$ and $D(\lambda)$ zero for purely positive imaginary λ to get square integrable behavior at ∞. This corresponds to bound state. Using (35) and (36) it can be shown that the bound state eigenvalue is again given by (33). Similarly it can also be shown that for purely negative imaginary λ it is reasonable to set the coefficients $A(\lambda)$ and $C(\lambda)$ zero in order to get bound state solution.

VI. IMPLICATION OF SCALING ANOMALY

As pointed out in section II, we now discuss the classical scale symmetry and scaling anomaly of (12). To do that, we rewrite (12) in the form of time independent Schrödinger equation

$$\mathcal{H}_r F_s(r) = \mathcal{E} F_s(r)\,, \tag{37}$$

where the Hamiltonian $\mathcal{H}_r = -\partial_r^2 - (\nu^2 - 1/4)/r^2$, $\nu = m - eQ\alpha - 1/2$ and the eigen-value $\mathcal{E} = E^2 - M^2$. Due to the cylindrical symmetry we just consider the problem on x-y plane, by setting $p_z = 0$. This is a well known one dimensional inverse square problem, discussed in many areas of physics, from molecular physics to black hole [16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. Classically (37) is scale covariant. The scale transformation $r \to \xi r$ and $t \to \xi^2 t$ (ξ is scaling factor) transforms the the Hamiltonian $\mathcal{H}_r \to (1/\xi^2)\mathcal{H}_r$. Scale covariance of $\mathcal{H}_r$ implies that it should not have any bound state. But, it is well known that there exists a 1-parameter family self-adjoint extensions (SAE) of $\mathcal{H}_r$ and due to this SAE, a single bound state is formed. The bound state energy for $\nu^2 \in [0, 1)$ is given by

$$\mathcal{E} = E^2 - M^2 = -\sqrt[\nu]{\frac{\sin(\Sigma/2 + 3\pi\nu/4)}{\sin(\Sigma/2 + \pi\nu/2)}}\,, \tag{38}$$

where Σ is the self-adjoint extension parameter for $\mathcal{H}_r$. Existence of this bound state immediately breaks the scale symmetry, which leads to scaling anomaly.

If we assume that the energy of the Dirac particle E to be real, then from the bound state condition $\mathcal{E} < 0$, we get a bound on the energy of the Dirac particle to be $E \in (-M, M)$. This bound still holds for the free Dirac particle, which we get by taking limit $\alpha \to 0$ in (38).

Quantum mechanically, scale transformation is associated with a generator, called scaling operator $\Lambda = \frac{1}{2}(rp_r + p_r r)$, where $p_r = -i\frac{d}{dr}$. It can be shown that the action of the operator Λ on a generic element of the self adjoint domain $D_\Sigma(\mathcal{H}_r)$ does throw the element outside the domain for some values of the self-adjoint extension parameter Σ. This indicates that there is scaling anomaly, occurred due to self-adjoint extensions. However, it can be shown that for $\Sigma = -\nu\pi/2$ and $\Sigma = -3\nu\pi/2$, the action of the operator Λ on any element of the domain $D_\Sigma(\mathcal{H}_r)$ does not throw it outside the domain. So, in these two cases scaling symmetry is still restored even after self-adjoint extensions and thus the bound state does not occur.

VII. CONCLUSION AND DISCUSSION

In this paper we calculated the solution of the Dirac equation in the field of an infinitely long solenoid. We showed that there is nontrivial bound state and as well as scattering state solution in the range $\nu^2 \in [0, 1)$. We showed only spin up particle state details of self-adjoint extensions in our calculation, but for all other spinors calculations are similar. We point out that for $\nu = 0$, the solution of $H^\dagger\phi^\pm = \pm iM\phi^\pm$ involves $r^{\frac{1}{2}}H_1^1(i\lambda_1 r)$, which

is not square integrable at the origin. So at $\nu = 0$ deficiency indices are zero, i.e, $n_+ = n_- = 0$, that means closure of domain is the self-adjoint extension for $\nu = 0$. We have projected the whole problem from 4-component Hilbert space $(L_2(drd\phi dz;\mathcal{C}^4))$ to 1-component Hilbert space $(L_2(drd\phi dz;\mathcal{C}))$ keeping in mind that we make the whole (4×4) Dirac Hamiltonian self-adjoint unlike Ref. [6], where they discussed the self-adjoint issue projecting the problem on $L_2(rdrd\phi dz;\mathcal{C}^2)$. The significant difference between these two approaches is that in our case we have a 1-parameter family of self-adjoint extensions unlike Ref. [6] where they have 2-parameter family of self-adjoint extensions. As in Ref. [16, 17] we studied that there is a relation between scattering state and bound state and it occurs for the purely imaginary value of λ. This has been done by demanding square integrability of the scattering state at spatial infinity. We showed that scattering state eigenvalue reduces to bound state eigenvalue for purely imaginary λ. Finally we discussed the implications of scaling anomaly on Dirac particle in background solenoid field. The energy of the Dirac particle in solenoid field background has bounds from both sides $E \in (-M, M)$. By taking appropriate limit, these bounds were also shown to hold for free Dirac particles.

VIII. ACKNOWLEDGMENT

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